Chapter 20 – Revision of Chapters 1-19

Solutions to 20A Short-answer questions

1
$$\begin{aligned} 2x + 3(4 - x) &= 8 \\ 2x + 12 - 3x &= 8 \\ -x &= -4 \\ x &= 4. \end{aligned}$$

2
$$\begin{aligned} \frac{at + b}{ct + d} &= 2 \\ at + b &= 2ct + 2d \\ (a - 2c)t &= 2d - b \\ t &= \frac{2d - b}{a - 2c}. \end{aligned}$$

3
$$\begin{aligned} \frac{4x}{3} - 4 &\le 2x - 3 \\ -4 + 3 &\le 2x - \frac{4x}{3} \\ -1 &\le \frac{2x}{3} \\ -3 &\le 2x \\ x &\ge -\frac{3}{2}. \end{aligned}$$

4 **a** For $x - y$ to be as small as possible choose the smallest possible value of x and the largest possible value of y.
Thus take $x = -4$ and $y = 8$.
Hence the smallest value of $x - y$ is -12.

b The largest possible value of $\frac{x}{y}$ is achieved by making x as large as possible and y as small as possible.
Thus take $x = 6$ and $y = 2$.
Hence the largest value of $\frac{x}{y}$ is 3.

c The largest possible value of $x^2 + y^2$ is obtained by choosing values of the largest magnitude for both x and y.
Thus take $x = 6$ and $y = 8$.
Hence the largest possible value of $x^2 + y^2$ is 100.

5 Let x be the number of the first type book and y be the number of the other type of book.
There is a total of 20 books. So
$x + y = 20 \qquad (1)$
There is total cost of \$720. So
$72x + 24y = 720 \qquad (2)$
Multiply equation (1) by 24.
$24x + 24y = 480 \qquad (3)$
Subtract equation (3) from equation (2).
$48x = 240$
$x = 5.$
Hence $x = 5$ and $y = 15$.
There were 5 of one type of book and 15 of the other.

6
$$\begin{aligned} \frac{1 - 5x}{3} &\ge -12 \\ 1 - 5x &\ge -36 \\ -5x &\ge -37 \\ x &\le \frac{37}{5}. \end{aligned}$$

7 $a = \dfrac{y^2 - xz}{10}$
When $x = -5, y = 7$ and $z = 6$,
$$\begin{aligned} a &= \frac{7^2 + 5 \times 6}{10} \\ &= \frac{79}{10}. \end{aligned}$$

8 **a** Midpoint $M(xy)$: $x = \dfrac{8+a}{2}$ and $y = \dfrac{14+b}{2}$

b If (5, 10) is the midpoint, $\dfrac{8+a}{2} = 5$ and $\dfrac{14+b}{2} = 10$. Hence $a = 2$ and $b = 6$.

9 **a** The line passes through $A(-2, 6)$ and $B(10, 15)$.

Using the form $y - y_1 = m(x - x_1)$,

$$m = \frac{15-6}{10-(-2)} = \frac{9}{12} = \frac{3}{4}$$

The equation is thus,

$$y - 6 = \frac{3}{4}(x+2)$$

Simplifying,

$$4y - 24 = 3x + 6$$

$$4y - 3x = 30.$$

b When $x = 0, y = \dfrac{15}{2}$

When $y = 0,\ x = -10$

By Pythagoras's theorem,

$$\text{the length of } PQ = \sqrt{(-10-0)^2 + \left(0 - \frac{15}{2}\right)^2} = \sqrt{100 + \left(-\frac{225}{4}\right)} = \sqrt{\frac{625}{4}} = \frac{25}{2}.$$

10 **a** $A = (-7, 6)$ and $B = (11, -5)$.

The midpoint $M(x, y)$ of AB has coordinates,

$x = \dfrac{-7+11}{2}$ and $y = \dfrac{6+(-5)}{2}$

The midpoint is $M(2, \frac{1}{2})$.

b The distance between A and B

$$= \sqrt{(11-(-7)) + (-5-6)} = \sqrt{18^2 + 11^2} = \sqrt{324 + 121} = \sqrt{445}$$

c The equation of AB.

Gradient, $m = \dfrac{-5-6}{11-(-7)} = -\dfrac{11}{18}$.

Using the form $y - y_1 = m(x - x_1)$.

$$y - 6 = -\frac{11}{18}(x+7)$$

Simplifying,

$$18y - 108 = -11x - 77.$$

$$18y + 11x = 31.$$

d The gradient of a line perpendicular to line AB is $\dfrac{18}{11}$.

The midpoint of AB is $M\left(2, \dfrac{1}{2}\right)$.

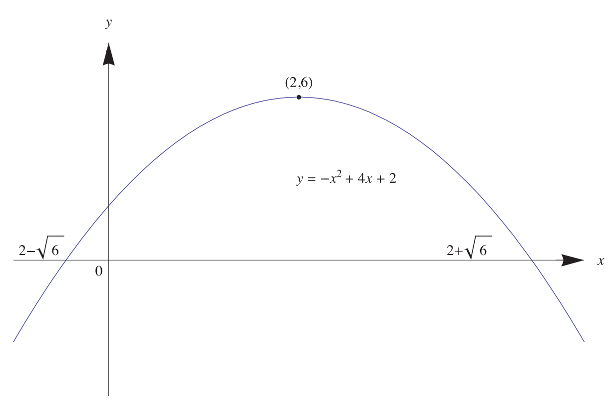
Using the form $y - y_1 = m(x - x_1)$.

$$y - \frac{1}{2} = \frac{18}{11}(x-2)$$

$$22y - 11 = 36x - 72$$

$$22y - 36x + 61 = 0.$$

11



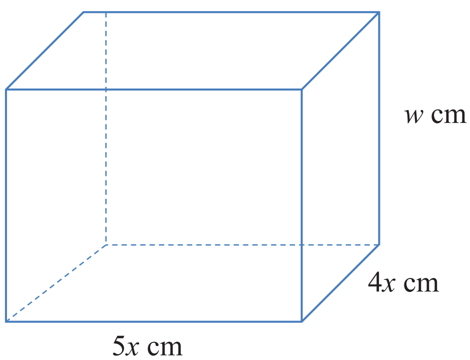
12 A parabola has turning point $(2, -6)$.
It has equation of the form
$y = k(x-2)^2 - 6$.
It passes through the point $(6, 12)$.
Hence,

$$12 = k(4)^2 - 6$$
$$18 = 16k$$
$$k = \frac{9}{8}.$$

Hence the equation is $y = \frac{9}{8}(x-2)^2 - 6$.

13 Let $P(x) = ax^3 + 4x^2 + 3$. It has remainder 3 when divided by $x - 2$.
The remainder theorem gives us that:
$P(2) = 3$.
That is, $3 = 8a + 16 + 3$.
Hence $a = -2$.

14



a The length of the wire is 6000 cm.
We have:

$$4 \times 5x + 4 \times 4x + 4w = 6000$$
$$5x + 4x + w = 1500$$
$$w = 1500 - 9x.$$

b Let V cm^3 be the volume of cuboid.

$$V = 5x \times 4x \times x$$
$$= 20x^2(1500 - 9x)$$

c We have $0 \le x \le \frac{500}{3}$ since $w = 1500 - 9x > 0$.

d

$$\text{If } x = 100,\ V = 20 \times 100^2(1500 - 9 \times 100)$$
$$= 200\,000 \times 600$$
$$= 120\,000\,000 \text{ cm}^3$$

15 Let n = number of square lino tiles
Let l = side length of the tile used

$$n \propto \frac{1}{l^2}$$
$$n = \frac{k}{l^2}$$

When $n = 900$, $l = 0.5$

$$900 = \frac{k}{0.5^2}$$
$$k = 225$$

Thus, the number of tiles with side length 0.75 m required is $\frac{225}{0.75^2} = 400$.

16 **a** Probability of both red $= \frac{4}{9} \times \frac{4}{9} = \frac{16}{81}$.

b Probability of both red
$= \frac{4}{9} \times \frac{7}{17} = \frac{28}{153}$.

17

	Box 1		
Box 2	1	3	5
2	3	5	7
4	5	7	9
6	7	9	11

Sample space = $\{3, 5, 7, 9, 11\}$
The outcomes are not equally likely.
Pr(divisible by 3) $= \frac{1}{3}$.

18 There are six letters and three vowels.

a The probability that the letter withdrawn is a vowel $= \frac{1}{2}$.

b The probability that the letter is a vowel is $\frac{1}{3}$.

19 This can be done simply by considering the cases.
SSF or SFF are the only two possibilities.
The probability of fruit on Wednesday

$$= 0.4 \times 0.6 + 0.6 \times 0.3$$
$$= 0.24 + 0.18$$
$$= 0.42.$$

20

Solve $\cos(3x)$

$$= \frac{1}{2} \text{ for } x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$3x = \ldots, -\frac{7\pi}{3}, -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots$$

$$x = -\frac{\pi}{9}, \frac{\pi}{9}.$$

21 The graph of $y = ax^3 + bx + c$ has intercepts (0,6) and $(-2, 0)$ and has a stationary point where $x = 1$.
$\frac{dy}{dx} = 3ax^2 + b.$

a The graph passes through (0,6). Therefore $6 = c$.

b The graph passes through $(-2, 0)$. Therefore

$$0 = -8a - 2b + 6 \qquad (1)$$

There is a stationary point where $x = 1$. Therefore

$$0 = 3a + b \qquad (2)$$

c Multiply (2) by 2.

$$0 = 6a + 2b \qquad (3)$$

Add equations (1) and (3).

$$0 = -2a + 6$$

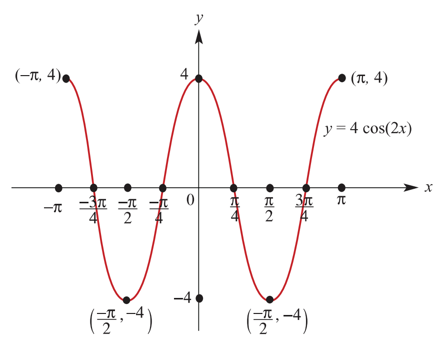
Therefore $a = 3$. Substitute in (2) to find $b = -9$.

22 $y = x^4$ and so $\frac{dy}{dx} = 4x^3$.
The gradient of the line $y = -32x + a$ is -32.
$4x^3 = -32$ implies $x^3 = -8$.
Hence $x = -2$
For $y = x^4$, when $x = -2$, $y = 16$.
Therefore for the tangent $y = -32x + a$,
$16 = -32 \times (-2) + a$.
Hence $a = -48$.

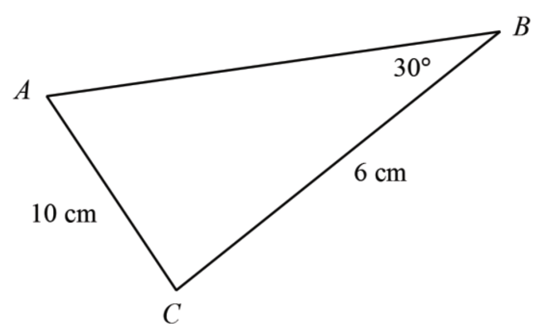
23 $f\colon [-\pi, \pi] \to R, f(x) = 4\cos(2x)$.

a Period $= \frac{2\pi}{2} = \pi$
Amplitude $= 4$

b

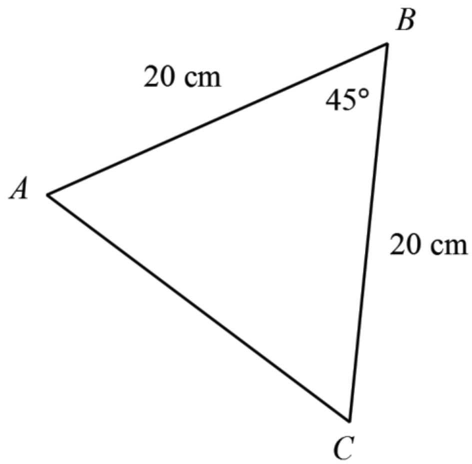


24 **a** Draw a diagram:



$$\frac{\sin(\angle BAC)}{6} = \frac{\sin(30°)}{10}$$
$$\sin(\angle BAC) = \frac{6\sin(30°)}{10} = 0.3$$

b Draw a diagram:



$$AC^2 = 20^2 + 20^2 - 2 \times 20 \times 20\cos(45°) = 800 - 400\sqrt{2}$$

25 a $A = \frac{1}{2}r^2\theta$

$$12 = \frac{1}{2} \times 6^2 \times \theta$$
$$\theta = \frac{2}{3}$$

b $DC = r_{OD}\theta$

$$= (6+3) \times \frac{2}{3} = 6$$

$$BA = r_{OB}\theta = 6 \times \frac{2}{3} = 4$$

$$\text{Perimeter} = BD + DC + CA + AB = 3 + 6 + 3 + 4 = 16\,\text{cm}$$

$$A_{OCD} = \frac{1}{2}r_{OD}^2\theta = \frac{1}{2} \times 9^2 \times \frac{2}{3} = 27\,\text{cm}^2$$

$$A_{OAB} = \frac{1}{2}r_{OB}^2\theta = \frac{1}{2} \times 6^2 \times \frac{2}{3} = 12\,\text{cm}^2$$

Thus,

$$A_{ABDC} = 27 - 12 = 15\,\text{cm}^2$$

26 a The first ball can be any ball except 1. The probability of a 3, 5 or 7 is $\frac{3}{4}$. There are 3 balls left and the probability of obtaining the white ball is $\frac{1}{3}$.
The probability of white on the second $= \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$.

b The sum of 8 can be obtained from the following ordered pairs: (1, 7), (7, 1), (3, 5), (5, 3).
The probability of each of these pairs $= \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.
Therefore the probability of obtaining a sum of 8 $= 4 \times \frac{1}{12} = \frac{1}{3}$

c We can see that for a sum of 8 we must only consider the pairs (1, 7), (7, 1), (3, 5), (5,3). The probability that the second is 1 is $\frac{1}{4}$.

27 The line $y = x + 1$ cuts the circle $x^2 + y^2 + 2x - 4y + 1 = 0$ at the points A and B.

To find the points of intersection:

$y = x + 1 \qquad (1)$

$x^2 + y^2 + 2x - 4y + 1 = 0 \qquad (2)$

Substitute from (1) into (2)

$x^2 + (x + 1)^2 + 2x - 4(x + 1) + 1 = 0$

$x^2 + x^2 + 2x + 1 + 2x - 4x - 4 + 1 = 0$

$2x^2 - 2 = 0$

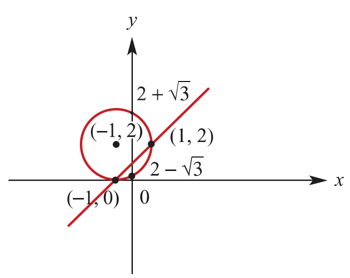
$2(x^2 - 1) = 0$

$x = 1$ or $x = -1$

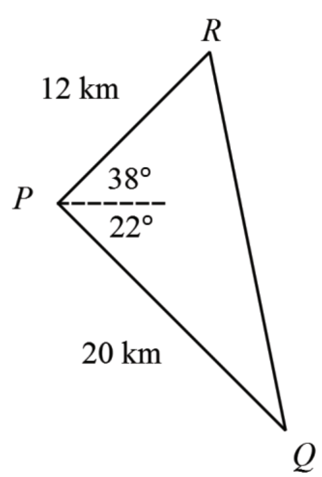
The points of intersection are $A(1, 2)$ and $B(-1, 0)$.

a The midpoint of $AB = (0, 1)$.

b



28 Draw a diagram:



$PQ^2 = 12^2 + 20^2 - 2 \times 12 \times 20 \cos(38° + 22°)$

$PQ = 4\sqrt{19}$ km

The distance between the two ships is $4\sqrt{19}$ km.

29 **a** $4^x - 5 \times 2^x - 24 = 0$.

Let $a = 2^x$.

The equation becomes.

$a^2 - 5a - 24 = 0$

$(a - 8)(a + 3) = 0$

$a = 8$ or $a = -3$.

Now $2^x > 0$ for all x and so there are no solutions of $2^x = -3$.

$2^x = 8$ implies $x = 3$.

b $2^{5-3x} = -4^{x^2} = 0$

$2^{5-3x} = 2^{2x^2}$

We note that if $2^a = 2^b$ then $a = b$.

Hence,

$5 - 3x = 2x^2$

$2x^2 + 3x - 5 = 0$

$(2x + 5)(x - 1) = 0.$

So, $x = -\dfrac{5}{2}$ or $x = 1$.

30 $a = 1,\ d = 5$

To find out how many terms there are, we need to solve:

$61 = 1 + 5(n - 1)$

Thus, $n = 13$terms.

$S_{13} = \dfrac{13}{2}(2 \times 1 + (13 - 1) \times 5)$

$= 403$

31 $a = x^2,\ r = x$

$\dfrac{a}{1 - r} = \dfrac{1}{6}$

$\dfrac{x^2}{1 - x} = \dfrac{1}{6}$

$6x^2 = 1 - x$

$6x^2 + x - 1 = 0$

$(2x + 1)(3x - 1) = 0$

$x = -\dfrac{1}{2}$ or $\dfrac{1}{3}$

32 $\dfrac{dy}{dx} = -4x + k$, where k is a constant.
Stationary point at $(1, 5)$.
$\dfrac{dy}{dx} = 0$ when $x = 1$.

$$0 = -4 + k$$
$$k = 4$$

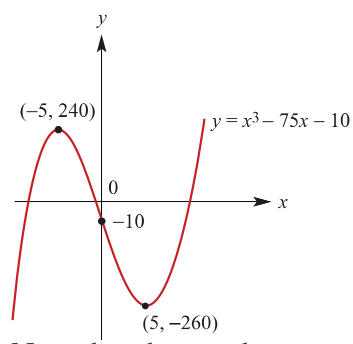
Thus,
$$\frac{dy}{dx} = -4x + 4$$
Integrating with respect to x.
$$y = -2x^2 + 4x + c.$$
When $x = 1$, $y = 5$.
Hence,
$$5 = -2 + 4 + c$$
$$c = 3.$$
The equation is $y = -2x^2 + 4x + 3$.

33 $y = ax^3 - 2x^2 - x + 7$ has a gradient of 4, when $x = -1$.
$\dfrac{dy}{dx} = 3ax^2 - 4x - 1.$
Therefore,
$$4 = 3a + 4 - 1$$
$$a = \frac{1}{3}.$$

34 Let $P(x) = 3x^2 + x + 10$.
$P(-b) = 3b^2 - b + 10$ and
$P(2b) = 12b^2 + 2b + 10$.
By the remainder theorem,
$$3b^2 - b + 10 = 12b^2 + 2b + 10$$
$$9b^2 + 3b = 0$$
$$3b(3b + 1) = 0.$$
Hence, $b = -\dfrac{1}{3}$.

35 $y = x^3$ and $y = x^3 + x^2 + 6x + 9$
The curves meet where
$x^3 = x^3 + x^2 + 6x + 9$.
That is,
$0 = (x + 3)^2$
Thus $x = -3$ and $y = -27$.
For the first curve,
$\dfrac{dy}{dx} = 3x^2$
and the second,
$\dfrac{dy}{dx} = 3x^2 + 2x + 6.$
When $x = -3$, $\dfrac{dy}{dx} = 27$ for the first curve.
When $x = -3$, $\dfrac{dy}{dx} = 27$ for the second curve.
There is a common tangent to the two curves.

36 **a** $y = x^3 - 75x - 10$
$\dfrac{dy}{dx} = 3x^2 - 75 = 3(x^2 - 25).$
Stationary points occur when $x = 5$ or $x \neq -5$



Note that the graph crosses the y-axis at -10.

b $x^3 - 75x - 10 = p$ has more than one real solution when the line with equation $y = p$ crosses the graph of $y = x^3 - 75x - 10$ more than once.
This is true when $-260 \le p \le 240$.

37 **a** Maximal domain $\mathbb{R}\backslash\{3\}$.

b Maximal domain $\mathbb{R}\backslash\{2\}$.

c Maximal domain $(-\infty,\ 2]$

d Maximal domain $[4, \infty)$

e Maximal domain $(-\infty,\ 5)$

Solutions to 20B Multiple-choice questions

1 B Period $= 2\pi \div \frac{1}{4} = 8\pi$
Amplitude $= 5$

2 A $f(x) = x^2 + 2x$
Average rate of change for the interval $[0, 3] = \frac{f(3) - f(0)}{3 - 0}$
$= \frac{15 - 0}{3}$
$= 5$

3 D $f\colon [1, 4) \to R, f(x) = (x - 2)^2 + 3$.
End points: $f(1) = 4$ and $f(4) = 7$.
The minimum value $= f(2) = 3$.
The range $= [3, 7)$.

4 D A function g with domain R has the following properties:

- $g'(x) = 3x^2 - 4x$
- the graph of $y = g(x)$ passes through the point $(1, 0)$

Taking the anti-derivative of g with respect to x:
$g(x) = x^3 - 2x + c$.
Also $g(1) = 0$,
so $0 = -1 + c$

$c = 1$.
Hence $g(x) = x^3 - 2x + 1$.

5 D Simultaneous equations

$$(m - 2)x + y = 0 \qquad (1)$$

$$2x + (m - 3)y = 0 \qquad (2)$$

Gradient of line (1) $= 2 - m$
Gradient of line (2) $= \frac{2}{3 - m}$
Infinitely many solutions implies
$2 - m = \frac{2}{3 - m}$.

Hence $(2 - m)(3 - m) = 2$

$$6 - 5m + m^2 = 2$$

$$m^2 - 5m + 4 = 0$$

$$(m - 1)(m - 4) = 0$$

$$m = 4 \text{ or } m = 1.$$

Both lines go through the origin and so there are infinitely many solutions for $m = 4$ or $m = 1$.

6 C $f(x) = 2\log_e(3x)$. If $f(5x) = \log_e(y)$
First $f(5x) = 2\log_e(15x)$.
Hence $2\log_e(15x) = \log_e(y)$
Thus $y = 15x^2 = 225x^2$.

7 C A bag contains 2 white balls and 4 black balls. Three balls are drawn from the bag without replacement.
Probability of black on the first $= \frac{2}{3}$.
There are now 2 white balls and 3 black balls.
Probability black on the second $= \frac{3}{5}$.
There are now 2 white balls and 2 black balls.
Probability of black on the third $= \frac{1}{2}$.
Probability of three black
$= \frac{2}{3} \times \frac{3}{5} \times \frac{1}{2} = \frac{1}{5}$

8 A $f\colon R \to R, f(x) = \frac{1}{3}x^3 - 2x^2 + 1$
$f'(x) = x^2 - 4x = x(x - 4)$
$f'(x) < 0$ if and only if $x(x - 4) < 0$.
This happens when the factors have different signs:
So either,
$x < 0$ and $x > 4$ or $x > 0$ and $x < 4$.
Only the second of these is possible.
Hence $0 < x < 4$.
This can also be seen by looking at the graph of $y = f'(x)$.

9 B $f(x) = \sqrt{2x+1}$ is defined for $2x + 1 \geq 0$.
That is the maximal domain of f is $x \geq -\frac{1}{2}$.
In interval notation $[-\frac{1}{2}, \infty)$.

10 A In algebraic notation, 11 is four times 9 more than x is written as $11 = 4(x + 9)$

11 B Time taken by the car $= \frac{120}{a}$ hours.
Time taken by the train $= \frac{120}{a-4}$.
Time taken by the train = time taken by the car +1.
Therefore,
$$\frac{120}{a-4} = \frac{120}{a} + 41$$
Multiplying both sides of the equation by $a(a - 4)$ we have,
$$120a = 120(a-4) + a(a-4)$$
$$120a = 120a - 480 + a^2 - 4a$$
$$0 = a^2 - 4a - 480$$
$$0 = (a-24)(a+20)$$
Therefore $a = 24$ or $a = -20$.
But we assume positive speed.

12 A The parabola that passes through the point $(-3, 12)$ and has its vertex at $(-2, 8)$ has equation of the form:
$y = k(x + 2)^2 + 8$.
It passes through the point $(-3, 12)$.
Hence,
$$12 = k(-1)^2 + 8$$
$$k = 4.$$
The equation is
$y = 4(x + 2)^2 + 8$.

13 E $\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-c^2}$
But θ is acute so $\sin\theta > 0$ and
$$\sin\theta = \sqrt{1-c^2}$$
$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ &= 2\sqrt{1-c^2} \times c\\ &= 2c\sqrt{1-c^2}\end{aligned}$$

14 A $f: [-3, 5) \to R,\ f(x) = 5 - 2x$.
The graph of f is a straight line with negative gradient.
$f(-3) = 11$ and $f(5) = -5$.
The range is $(-5, 11]$

15 E $f: [-3, 2) \to R,\ f(x) = 2x^2 + 7$.
The graph is a parabola with a minimum at $(0, 7)$.
$f(-3) = 25$ and $f(2) = 15$.
The range is $[7, 25]$

16 A
$$V = kr^3$$
$$8 = k(1)^3$$
$$k = 8$$
$$V = 8r^3$$
$$a = 8(2)^3$$
$$= 64$$
$$V = 8r^3$$
$$16 = 8b^3$$
$$b^3 = 2$$
$$b = 2^{\frac{1}{3}}$$

17 B
$$\begin{aligned}\frac{\sqrt{0.144 \times 10^5}}{2 \times 10^4} &= \frac{\sqrt{1.44 \times 10^4}}{2 \times 10^4}\\ &= \frac{\sqrt{1.2^2} \times \sqrt{10^4}}{2 \times 10^4}\\ &= \frac{1.2 \times 10^2}{2 \times 10^4}\\ &= 0.6 \times 10^{-2}\\ &= 6 \times 10^{-3}\end{aligned}$$

18 A
$$F = \frac{k}{R^2}$$
$$1400 = \frac{k}{7^2}$$
$$k = 68\,600$$
$$F = 68\,600R^{-2}$$

19 B $x + 1$ is a factor of $x^2 + ax + b$, then $-a + b + 7$ equals?
By the factor theorem,
$1 - a + b = 0$.
Thus $-a + b = -1$
$$-a + b + 7 = -1 + 7 = 6$$

20 A The choices are all cubic functions of the form $y = a(x + h)^3 + b$ where $a < 0$.
The graph shows a stationary point of inflexion at $(-1, 2)$
Hence it is of the form
$y = a(x + 1)^3 + 2$.
It passes through the origin.

21 B $f : R \to R,\ f(x) = x$.
$f(x) - f(-x) = x - (-x) = 2x$.

22 C $a = 4,\ d = 5$
$$S_{10} = \frac{n}{2}(2a + (n - 1)d)$$
$$= \frac{10}{2}(2 \times 4 + 9 \times 5)$$
$$= 265$$

23 C
$$t_n = ar^{n-1} = 3 \times 4^n$$
$$t_{20} = 3 \times 4^{19}$$

24 D
$$t_n = a + (n - 1)d$$
$$144 = 9 + 15 \times d$$
$$d = 9$$

25 D Whole function being multiplied by 3: dilation from x-axis by factor of 3
Coefficient of x is 2: dilation from the y-axis by a factor of $\frac{1}{2}$

26 C $25^x - 7 \times 5^x + 12 = 0$.
Let $a = 5^x$
$a^2 - 7a + 12 = 0$
$(a - 3)(a - 4) = 0$
Hence $a = 3$ or $a = 4$.
That is $5^x = 3$ or $5^x = 4$.
Therefore,
$x = \log_5 3$ or $x = \log_5 4$

27 B A particle moves in a straight line so that its position s m relative to O at a time t seconds ($t > 0$), is given by $s = 4t^3 - 5t - 10$.
The velocity $\frac{ds}{dt} = v = 12t^2 - 5$.
The acceleration $= \frac{dv}{dt} = 24$.
When $t = 1$, the acceleration is 24 m/s^2.

28 A The average rate of change of the function $y = 2x^4 + x^3 - 1$ between $x = -1$ and $x = 1$ is equal to $\frac{2 - 0}{2} = 1$.

29 B
$$\frac{AB}{\sin(C)} = \frac{AC}{\sin(B)}$$
$$AB = \frac{12\sin(180^\circ - 100^\circ - 30^\circ)}{\sin(100^\circ)}$$
$$= \frac{12\sin(50^\circ)}{\sin(100^\circ)}$$

30 B
$$V = kw^2$$
$$500 = k(10)^2$$
$$k = 5$$

Hence,
$1125 = 5w^2$
$w = 15$

31 B $A = \frac{1}{2} \times 15 \times 10 \times \sin(30°)$
$= 37.5\,\text{cm}^2$

32 E A function $f: R \to R$ is such that

- $f'(x) = 0$ where $x = 3$
- $f'(x) = 0$ where $x = 5$
- $f'(x) > 0$ where $3 < x < 5$
- $f'(x) < 0$ where $x > 5$
- $f'(x) < 0$ where $x < 3$

Stationary points when $x = 3$ and $x = 5$.
Immediately to the left of 3, $f'(x) < 0$ and immediately to the right of 3, $f'(x) > 0$.
Therefore there is a local minimum at $x = 3$.
Immediately to the left of 5, $f'(x) > 0$ and immediately to the right of 5, $f'(x) < 0$.
Therefore there is a local maximum at $x = 5$.

33 E Number of ways that two girls can be selected from eight: ${}^8C_2$
Number of ways that two boys can be selected from 12: ${}^{12}C_2$
Hence, number of committees:
${}^8C_2 \times {}^{12}C_2$

34 D $(2x - 1)^5 = \sum_{i=0}^{5} {}^5C_i(2x)^i(-1)^{5-i}$
The coefficient of x^2 can be found by letting $i = 2$ in the expression
${}^5C_i(2)^i(-1)^{5-i}$
${}^5C_2(2)^2(-1)^{5-2} = -2^2 \times {}^5C_2$

35 D $l = \theta\frac{\pi}{180} \times r$
$7 = \frac{18\pi}{180} \times r$
$r = \frac{70}{\pi}$ mm

36 B The tangent at the point $(1, 5)$ on the graph of the curve $y = f(x)$ has equation $y = 4 + x$.
The tangent at the point $(3, 6)$ on the curve
The transformation is '2 to the right' and '1 up'.
So the tangent at the point (3, 6) on the curve
$y = f(x - 2) + 1$ is a translation of $y = 4 + x$.
It transforms to:
$y - 1 = 4 + x - 2$.
That is,
$y = 3 + x$.

37 D The graph of the of the derivative function f' of the cubic function with rule $y = f(x)$ crosses the x axis at $(1, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 12.
$f'(x) = k(x - 1)(x + 3)$.
The maximum value of the derivative function is 12. This occurs when $x = 2$.
This tells us that $k < 0$ as the quadratic has a maximum.
The turning points of the cubic occur at $x = 1$ and $x = -3$.
For a local maximum we look where the gradient changes from positive to

negative going from left to right.
$f'(x) = k(x - 1)(x + 3)$.
For $x < 1$, $f'(x) > 0$ (Remember $k < 0$)
For $x > 1$, $f'(x) < 0$.
Hence there is a local maximum where $x = 1$.

38 A Let $f'(x) = 5g'(x) + 4$ and $f(1) = 5$ and $g(x) = x^2 f(x)$.
We have $f(x) = 5g(x) + 4x + c$ and $f(1) = 5$ and $g(1) = f(1) = 5$
So $5 = 25 + 4 + c$.
Hence $c = -24$.
Finally, $f(x) = 5g(x) + 4x - 24$

39 C Identify the incorrect statement by checking each one at a time.
This can be done by inputting each one of the statements, one at a time, into the CAS calculator to verify if they are true or false.

40 A Start with $y = f(x)$
Reflection in the y-axis: replace $x$with $-x$ $(y = f(-x))$
Dilation of factor 2 from the x-axis: replace y with $\frac{y}{2}$($\frac{y}{2} = f(-x)$)
Translation of 2 units up: replace y with $y - 2$($\frac{y-2}{2} = f(-x)$)
$y = 2f(-x) + 2$

41 C $a = 1,\ r = \frac{1}{2}$

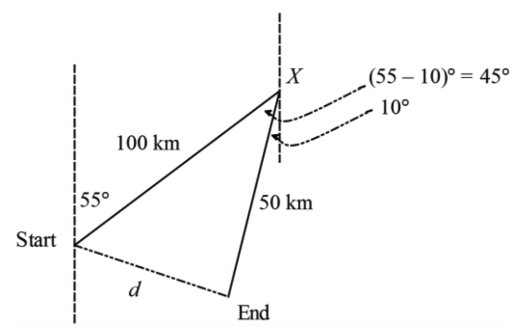
$$S = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$$

42 D

$$S_n = \frac{n}{2}(2a + (n-1)d)$$
$$100 = \frac{n}{2}(2 \times 1 + (n-1) \times 2)$$
$$200 = n(2 + 2n - 2)$$
$$200 = n(2n)$$
$$2n^2 = 200$$
$$n = 10$$

$$t_{10} = 1 + (10 - 1)2 = 19$$

43 D



Let Start = S and End = E.

$$d^2 = SX^2 + XE^2 - 2 \times SX \times XE \times \cos(\angle SXE)$$
$$= 100^2 + 50^2 - 2 \times 100 \times 50 \times \cos(45^\circ)$$
$$= 12\,500 - 5000\sqrt{2}$$
$$d \approx 74\,\text{km}$$

Solutions to 20C Extended-response questions

1 a i Gradient of $AB = \dfrac{16 - b^2 - 16}{b}$

$= -b$

ii $f'(x) = -2x$

The tangent at point $(x, f(x))$ has gradient $-2x$

$-2x = -b$

$x = \dfrac{b}{2}$

The tangent at the point where $x = \dfrac{b}{2}$ has gradient $-b$.

b i Area of a trapezium $= \dfrac{h}{2}(a + b)$

$S(b) = \dfrac{b}{2}(16 - b^2 + 16)$

$= \dfrac{b}{2}(32 - b^2)$

ii $\dfrac{b}{2}(32 - b^2) = 28$

$32b - b^3 = 56$

$b^3 - 32b + 56 = 0$

$(b - 2)(b^2 + 2b - 28) = 0$

$b = 2$ or $(b + 1)^2 = 29$

Thus $S(2) = 28$.

The other solutions of the equation are not in the interval $(0, 4)$.

The area of the trapezium is 28 when $b = 2$.

2 $f(x) = (\sqrt{x} - 2)^2(\sqrt{x} + 1)^2$

a To find the x-intercept

$f(x) = 0$ implies $\sqrt{x} - 2 = 0$ or $\sqrt{x} + 1 = 0$

Thus $\sqrt{x} - 2 = 0$ which implies $x = 4$.

Therefore $a = 4$.

b From the graph there is a local maximum at $x = \dfrac{1}{4}$ and a local minimum at $A(4, 0)$.

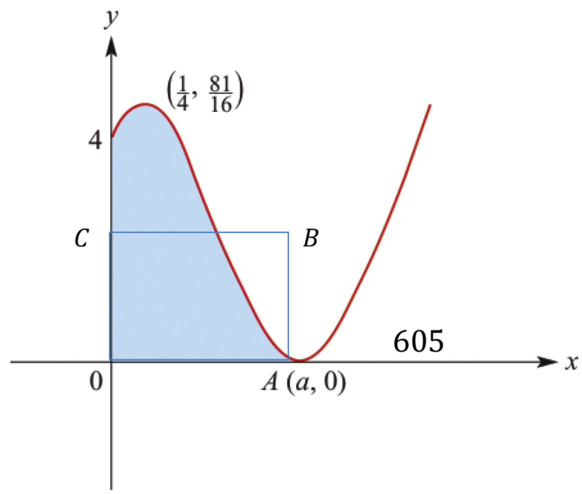
The graph has negative gradient for the interval $\left(\dfrac{1}{4}, 4\right)$

c $OABC$ is a rectangle with
$OA = BC = 4$.
Let $OC = AB = x$.

$$4x = \frac{136}{15}$$

$$x = \frac{34}{15}$$

That is, $OC = \dfrac{34}{15}$



3 **a** $v = 4t - 6$
Find an expression for x the position at time t by finding the antiderivative.
$x = 2t^2 - 6t + c$
When $t = 0$, $x = 0$ and therefore $c = 0$.
$x = 2t^2 - 6t$

b $x(3) = 18 - 18$
$= 0$
The body has returned to O after 3 seconds.

c The average velocity $= \dfrac{\text{change in position}}{\text{change in time}} = 0$ cm/s

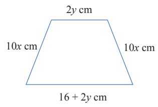
d Body reverses direction when $v = 0$.

$$4t - 6 = 0$$

$$t = \frac{3}{2}$$

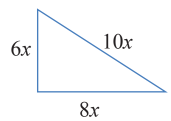
$x\left(\dfrac{3}{2}\right) = 2 \times \dfrac{9}{4} - 6 \times \dfrac{3}{2} = -\dfrac{9}{2}.$
The particle returns to O at $t = 3$.
The total distance travelled = 9 centimetres.

e Average speed $= \dfrac{9}{3} = 3$ centimetres/second.

4 

a $$16x + 2y + 2y + 10x + 10x = 52$$
$$36x + 4y = 52$$
$$4y = 52 - 36x$$
$$y = 13 - 9x$$

b Using Pythagoras' theorem.



Height$s$ = $6x$ cm

$$A(x) = \frac{6x}{2}(2y + 16x + 2y) \qquad \text{(Using the formula for the area of a trapezium)}$$
$$= 3x(4y + 16x)$$
$$= 3x(52 - 36x + 16x) \qquad \text{(Substituting for } y \text{ from part } \mathbf{a}\text{)}$$
$$= 156x - 60x^2$$

c Finding the derivative:
$A'(x) = 156 - 120x$
$A'(x) = 0$ implies $x = \frac{13}{10}$. A maximum occurs at this value as
$A(x) = 156x - 60x^2$ is a quadratic with negative coefficient of x^2.
Substituting for x in $y = 13 - 9x$ gives $y = \frac{13}{10}$.

5 **a** Total area of the two squares $= x^2 + y^2, \qquad x \le y$
Total length of fencing $= 2x + 3y$
Given that the length of the fencing is 5200 m
$$x + 3y = 5200$$
$$3y = 5200 - 2x$$
$$y = \frac{5200 - 2x}{3}$$

Therefore the total area, $\quad A = x^2 + \left(\frac{5200 - 2x}{3}\right)^2$

b $$A = x^2 + \frac{5200^2}{9} - \frac{20\,800x}{9} + \frac{4x^2}{9}$$
$$= \frac{13x^2}{9} - \frac{20\,800x}{9} + \frac{5200^2}{9}$$
$$A'(x) = \frac{26x}{9} - \frac{20\,800}{9}$$

$A'(x) = 0$ implies $x = 800$

The parabola has positive coefficient of x^2 and therefore a minimum when $x = 800$.

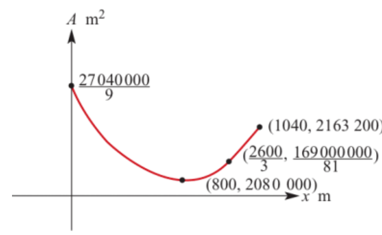
When $x = 800$ substituting in $y = \frac{5200 - 2x}{3}$ gives $y = 1200$.

Thus, substituting these values into the area formula gives a minimum area of $2\,080\,000\,\text{m}^2$

c $x \geq 0$ and $x \leq y$

Substitute $y = \frac{5200 - 2x}{3}$

$$\frac{5200 - 2x}{3} \geq x$$
$$5200 - 2x \geq 3x$$
$$5200 \geq 5x$$
$$x \leq 1040$$



6 $f(x) = -x^3 + ax^2$.

$f'(x) = -3x^2 + 2ax$

a **i** f has negative gradient for $f'(x) < 0$

$$-3x^2 + 2ax < 0$$
$$-x(3x - 2a) < 0$$
$$x < 0 \text{ or } x > \frac{2a}{3}$$

ii f has positive gradient for $f'(x) > 0$

$$0 < x < \frac{2a}{3}$$

b When $x = a$,

$$f'(a) = -3a^2 + 2a^2 \qquad \text{and} \qquad f(a) = 0$$
$$= -a^2$$

The equation of the tangent at the point $(a, f(a))$

$y - f(a) = f'(a)(x - a)$

Thus, $y = -a^2(x - a)$

c The gradient of the normal $= -\dfrac{1}{f'(a)}$

$= \dfrac{1}{a^2}$

The equation of the normal is

$$y = \frac{1}{a^2}(x - a)$$

7 a

(0, 1)	(0, 3)	(0, 5)	(0, 7)	(0, 9)	(0, 11)
(2, 1)	(2, 3)	(2, 5)	(2, 7)	(2, 9)	(2, 11)
(4, 1)	(4, 3)	(4, 5)	(4, 7)	(4, 9)	(4, 11)
(6, 1)	(6, 3)	(6, 5)	(6, 7)	(6, 9)	(6, 11)
(8, 1)	(8, 3)	(8, 5)	(8, 7)	(8, 9)	(8, 11)
(10, 1)	(10, 3)	(10, 5)	(10, 7)	(10, 9)	(10,11)

36 outcomes

b Table showing sums

	0	2	4	6	8	10
1	1	3	5	7	9	11
3	3	5	7	9	11	13
5	5	7	9	11	13	15
7	7	9	11	13	15	17
9	9	11	13	15	17	19
11	11	13	15	17	19	21

Let X be the sum of the results.

i $\Pr(X = 1) = \dfrac{1}{36}$

ii $\Pr(X = 13) = \dfrac{5}{36}$

iii $\Pr(X = 9) = \dfrac{5}{36}$

c $\Pr(X = 15|X > 7) = \dfrac{\Pr(X = 15)}{\Pr(X > 7)}$

$= \dfrac{2}{13}$

8 **a** **i** $a = 50\,000$

$d = 5000$

ii Let $t_n = 100\,000$

$$100\,000 = 50\,000 + (n-1) \times 5000$$
$$50\,000 = 5000(n-1)$$
$$10 = n - 1$$
$$n = 11$$

Production is doubled in the 11th month.

iii $S_{36} = \dfrac{36}{2}(2 \times 50\,000 + (36-1) \times 5000)$

$= 4\,950\,000$

In the first 36 months, 4 950 000 litres will be produced.

b **i** $a = 12\,000$

$r = 1.1$

ii $S_{12} = \dfrac{12\,000(1 - 1.1^{12})}{1 - 1.1}$

$= 256\,611.405207$

The total amount of drink produced in the first 12 months is 256 611 litres.

c Solve $12\,000 \times 1.1^{n-1} > 5000(n-1) + 50\,000$ which gives $n > 30.345$. Hence, production becomes faster in the 31st month.

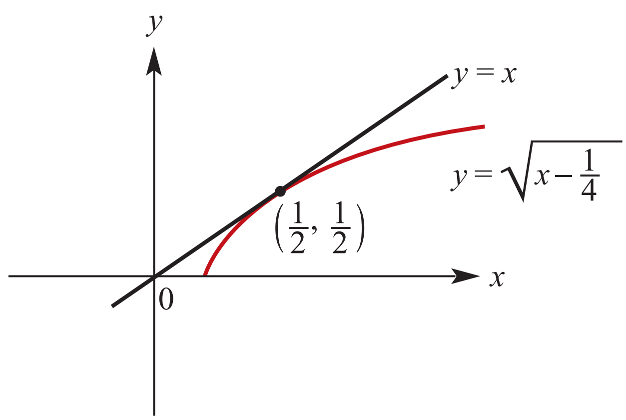
9 **a** For function to be defined $x - 2a \geq 0$. That is $x \geq 2a$.

b

$$\sqrt{x - 2a} = x$$
$$x - 2a = x^2 \qquad \text{Squaring both sides}$$
$$x^2 - x + 2a = 0$$
$$x^2 - x + \frac{1}{4} = -2a + \frac{1}{4} \qquad \text{Completing the square}$$
$$\left(x - \frac{1}{2}\right)^2 = \frac{-8a + 1}{4}$$
$$x = \frac{1}{2} + \frac{\sqrt{1 - 8a}}{2} \text{ or } x = \frac{1}{2} - \frac{\sqrt{1 - 8a}}{2}$$

c The equation $f(x) = x$ has one solution for $a = \dfrac{1}{8}$

d



10 a Probability that Frederick goes to the library on each of the next three nights

$= 0.7 \times 0.7 \times 0.7$

$= 0.343$

b The possible sequences for 3 days for exactly two days going to the library:

LSLL LLSL LLLS

Probability of exactly two nights

$= 0.3 \times 0.6 \times 0.7 + 0.7 \times 0.3 \times 0.6 + 0.7 \times 0.7 \times 0.3$

$= 0.126 + 0.126 + 0.147$

$= 0.399$

11 a Probability that sticks are brought from Platypus for the next three years

$= 0.75 \times 0.75 \times 0.75$

$= 0.4219$ (correct to four decimal places.)

b The possible sequences for three years for exactly two years buying from Platypus

PNPP PPNP PPPN

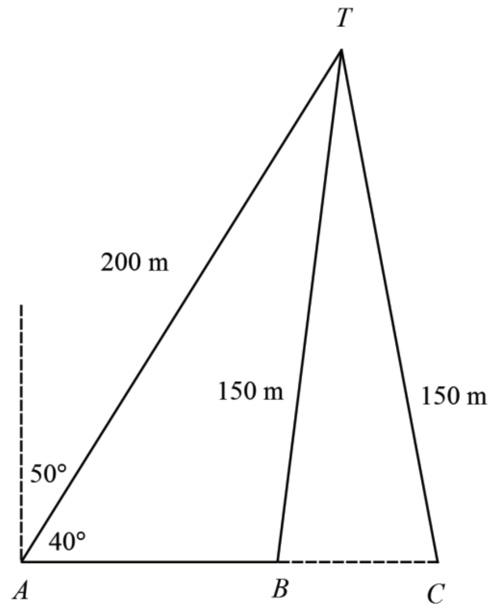
Probability of buying from Platypus for exactly two of the three years

$= 0.25 \times 0.2 \times 0.75 + 0.75 \times 0.25 \times 0.2 + 0.75 \times 0.75 \times 0.25$

$= 0.2156$ (correct to four decimal places.)

c Probability that they will buy from Platypus in the third year is 0.6125

12 Draw a diagram:



a $\dfrac{\sin(A)}{150} = \dfrac{\sin(\theta)}{200}$

$$\sin(\theta) = \frac{200\sin(40°)}{150}$$

$$\theta = \sin^{-1}\left(\frac{200\sin(40°)}{150}\right),\ 180° - \sin^{-1}\left(\frac{200\sin(40°)}{150}\right)$$

$$= 58.9869695348°,\ 121.013020465°$$

Thus,

$$\angle TBA = 121.013020465°$$

$$\angle TCA = 58.9869695348°$$

Hence,

$$\angle TBC = \angle TCB = 58.9869695348°$$

$$\angle BTC = 180° - 2 \times 58.9869695348°$$

$$= 62.0260609304°$$

Now use cosine rule to find BC

$$BC^2 = 150^2 + 150^2$$

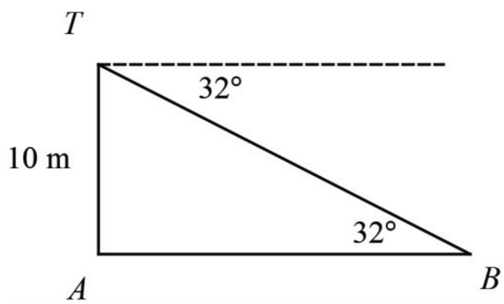
$$- 2 \times 150 \times 150 \times \cos(62.0260609304°)$$

$$= 23891.8542134$$

$$BC = 154.569900735$$

The distance between Brian and Colin is 155 metres.

b **i**

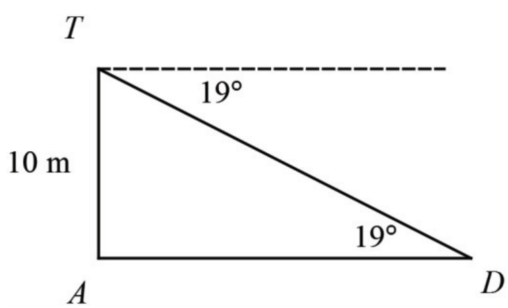


$$\tan(32^\circ) = \frac{10}{AB}$$

$$AB = \frac{10}{\tan(32^\circ)}$$

$$= 16.00 \text{ m (to 2 d.p.)}$$

ii

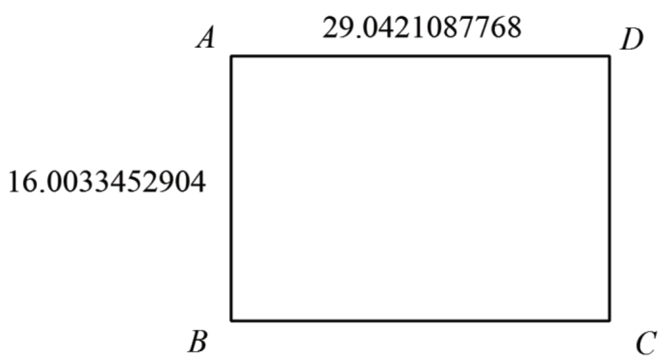


$$\tan(19^\circ) = \frac{10}{AD}$$

$$AD = \frac{10}{\tan(19^\circ)}$$

$$= 29.04 \text{ m (to 2 d.p.)}$$

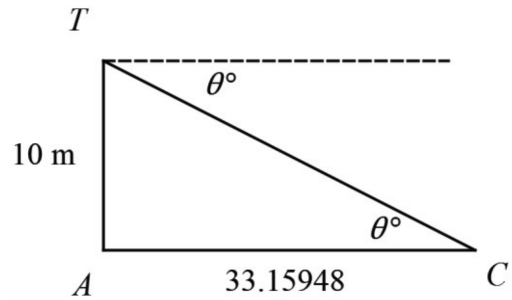
iii



$$AD = \sqrt{16.0033^2 + 29.0421^2}$$

$$= 33.15948$$

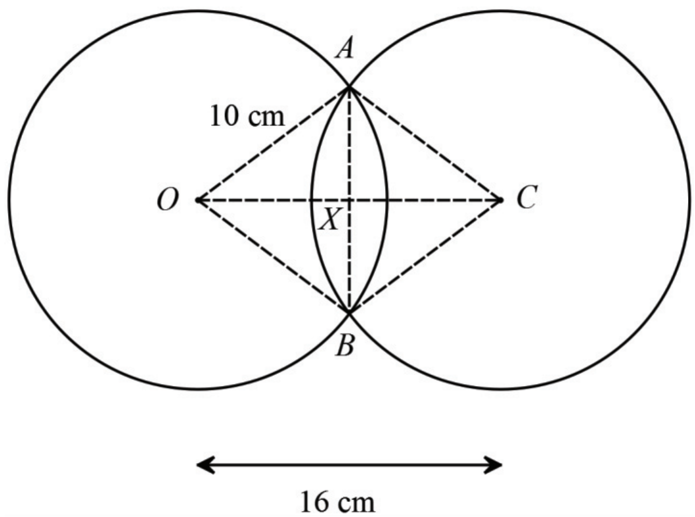
Now,



Thus,

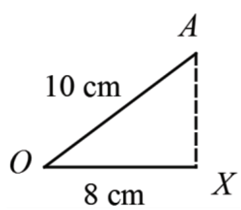
$$\theta = \tan^{-1}\left(\frac{10}{33.15948}\right)$$
$$= 17°$$

c



Need to find the angle $\angle AOB$

Consider triangle OAX:



$$\angle AOX = \cos^{-1}\left(\frac{8}{10}\right)$$

Hence, $\angle AOB = 2\cos^{-1}\left(\frac{8}{10}\right)$.

The minor segment cut off by the chord AB now be found:

$$A = \frac{1}{2}r^2(\theta - \sin(\theta))$$
$$= \frac{1}{2} \times 10^2 \times \left(2\cos^{-1}\left(\frac{8}{10}\right) - \sin\left[2\cos^{-1}\left(\frac{8}{10}\right)\right]\right)$$
$$= 16.3501108793$$

The total area is thus $16.35 \times 2 = 32.7\,\text{cm}^2$

13 **a** The line has negative gradient.
The range $= [-mb + 3,\ -ma + 3]$

b The coordinates of the midpoint are found by using the midpoint of the line segment joining (x_1, y_1) to $(x_2,\ y_2)$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
The midpoint of $AB = \left(\frac{a+b}{2}, \frac{-ma - mb + 6}{2}\right)$

c Gradient line through $AB = -m$
Gradient of a line perpendicular to $AB = \frac{1}{m}$

$$y - \left(\frac{-ma - mb + 6}{2}\right) = \frac{1}{m}\left(x - \left(\frac{a+b}{2}\right)\right)$$
$$2my + m^2a + m^2b - 6m = 2x - (a+b)$$
$$2my - 2x = -m^2(a+b) + 6m - (a+b)$$

d The transformation is defined by $(x, y) \to (x - 3, y + 5)$.
If $(x, y) \to (x', y')$ then
$x' = x - 3$ and $y' = y + 5$
Hence, $x = x' + 3$ and $y = y' - 5$
Substituting in $y = -mx + 3$ gives
$$y' - 5 = -m(x' + 3) + 3$$
$$y' = -mx' - 3m + 8$$
The equation of the image is $y = -mx - 3m + 8$
Considering the end points:
$(a, -ma + 3) \to (a - 3,\ -ma + 8)$
and
$(b, -mb + 3) \to (b - 3, -mb + 8)$.

e The transformation is defined by $(x, y) \to (-x, y)$.
If $(x, y) \to (x', y')$ then
$x' = -x$ and $y' = y$
The line is transformed to $y' = mx' + 3$.
That is, $y = mx + 3$

Considering the end points:
$(a, -ma + 3) \to (-a, -ma + 3)$
and
$(b, -mb + 3) \to (-b, -mb + 3)$.

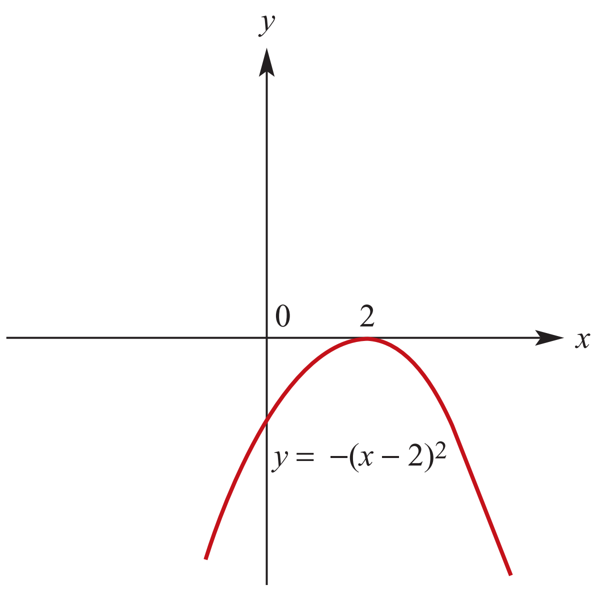
f If $a = 0$ the midpoint of AB has coordinates $\left(\frac{b}{2}, \frac{-mb + 6}{2}\right)$

Thus $\frac{b}{2} = 6$ and $\frac{-mb + 6}{2} = -4$

Hence $b = 12$ and $m = \frac{7}{6}$

14 a $f(x) = (p - 1)x^2 + 4x + (p - 4)$

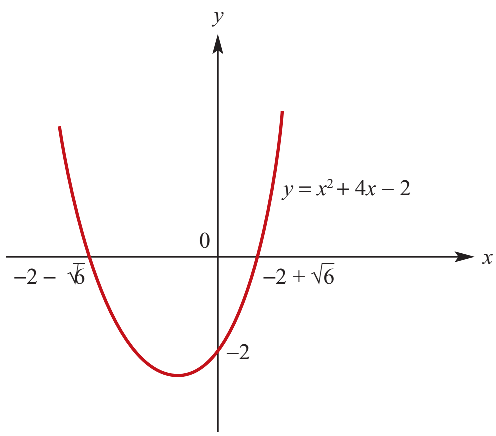
i When $p = 0$, $f(x) = -x^2 + 4x - 4 = -(x^2 - 4x + 4) = -(x - 2)^2$



ii When $p = 2$, $f(x) = x^2 + 4x - 2$

For the x axis intercepts:

$$x^2 + 4x - 2 = 0$$
$$x^2 + 4x + 4 = 6$$
$$(x + 2)^2 = 6$$
$$x = -2 + \sqrt{6} \text{ or } x = -2 - \sqrt{6}$$



b $f'(x) = 2(p-1)x + 4$

$f'(x) = 0$ implies $x = \dfrac{2}{1-p}$

and $f\left(\dfrac{2}{1-p}\right) = (p-1) \times \dfrac{4}{(1-p)^2} + \dfrac{8}{1-p} + (p-4)$

$$= \frac{4}{(p-1)} + \frac{8}{1-p} + (p-4)$$

$$= \frac{-4}{(p-1)} + (p-4)$$

$$= \frac{p^2-5p}{p-1}$$

The coordinates of the turning point are $\left(\dfrac{2}{1-p}, \dfrac{p^2-5p}{p-1}\right)$

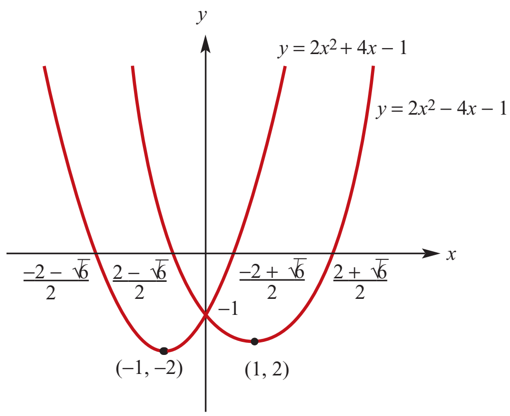
c The turning point lies on the x axis when the y coordinate is zero.
That is, when $5p - p^2 = 0$.
$p = 0$ or $p = 5$

d The discriminant of the quadratic $(p-1)x^2 + 4x + (p-4)$ is

$$16 - 4(p-1)(p-4) = 16 - 4[p^2 - 5p + 4]$$
$$= -4p^2 + 20p$$

There are two solutions when the discriminant is positive.
That is, when $-4p^2 + 20p > 0$
Equivalently when $p^2 - 5p < 0$
Thus $0 < p < 5$ and $p \neq 1$.

e The question should ask to sketch the graph of $y = g(x)$ and the graph of the reflection in the y-axis.



15 $h(t) = 2.3\cos(kt)$

a High tide occurs every 12 hours

$$\frac{2\pi}{k} = 12$$

$$k = \frac{\pi}{6}$$

b
$$\begin{aligned} h(1.5) &= 2.3\cos\left(\frac{\pi}{6} \times 1.5\right) \\ &= 2.3\cos\left(\frac{\pi}{4}\right) \\ &= 2.3 \times \frac{1}{\sqrt{2}} \end{aligned}$$

This is measured in metres

Thus the height of the road above mean sea level is:

$$\begin{aligned} 2.3 \times \frac{1}{\sqrt{2}} \text{ metres } &= 1.15\sqrt{2} \times 100 \text{ cm} \\ &= 115\sqrt{2} \text{ cm} \end{aligned}$$

c
$$\begin{aligned} h(1.5) &= 2.3\cos\left(\frac{\pi}{6}\right) \\ &= 2.3 \times \frac{\sqrt{3}}{2} \end{aligned}$$

Thus the height of the raised footpath above mean sea level is:

$$\begin{aligned} 2.3 \times \frac{\sqrt{3}}{2} \text{ metres } &= 1.15\sqrt{3} \times 100 \text{ cm} \\ &= 115\sqrt{3} \text{ cm} \end{aligned}$$